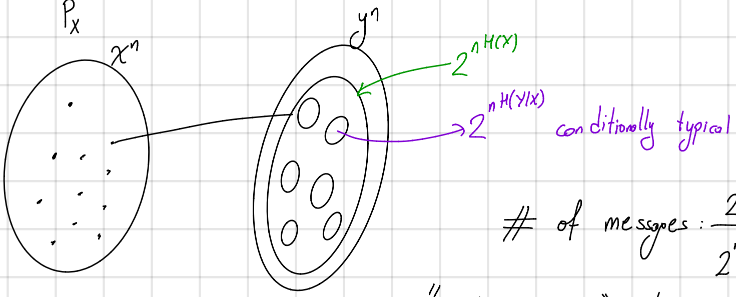
10/04/2016
Tuesday

$$C = \max_{P_X} I(X;Y)$$



$$\# \text{ of messages} : \frac{2^{nH(Y)}}{2^{nH(Y|X)}} = 2^{nI(X;Y)}$$

"Sphere Packing" idea.

Jointly typical:

$$A_\varepsilon^{(n)}(P_{XY}) = \left\{ (x^n, y^n) : \left| \frac{1}{n} \log \frac{1}{P_{X^nY^n}(x^n, y^n)} - H(X,Y) \right| \leq \varepsilon,\ X^n \in A_\varepsilon^{(n)},\ Y^n \in A_\varepsilon^{(n)} \right\}$$

Properties of $A_\varepsilon^{(n)}(P_{XY})$:

1. If X^n, Y^n jointly i.i.d. $\sim P_{XY}$

$$\mathbb{P}[(X^n, Y^n) \in A_\varepsilon^{(n)}] \longrightarrow 1 \quad \text{as} \quad n \to \infty$$

Union bound $\mathbb{P}(A \text{ or } B \text{ or } C) \leq \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C)$

2. $|A_\varepsilon^{(n)}| \in \left[(1-\varepsilon) 2^{n(H(X,Y) - \varepsilon)},\ 2^{n(H(X,Y)+\varepsilon)} \right]$

↳ lower bound is for n sufficiently large

3. If X^n iid $\sim P_X$, Y^n i.i.d. $\sim P_Y$, $X^n \perp\!\!\!\perp Y^n$

$$\mathbb{P}[(X^n, Y^n) \in A_\varepsilon^{(n)}] \in \left[(1-\varepsilon) 2^{-n(I(X;Y)+3\varepsilon)},\ 2^{-n(I(X;Y)-3\varepsilon)} \right]$$

↳ lower bound is for large enough n.

proof

$$\mathbb{P}[(X^n, Y^n) \in A_\varepsilon^{(n)}] = \sum_{x^n, y^n \in A_\varepsilon^{(n)}} P_{X^nY^n}(x^n, y^n) = \sum_{x^n, y^n \in A_\varepsilon^{(n)}} P_{X^n}(x^n) P_{Y^n}(y^n)$$

$$\leq |A_\varepsilon^{(n)}| \, 2^{-n(H(X)+H(Y)-2\varepsilon)} \leq 2^{-n(H(X)+H(Y)-H(X,Y)-3\varepsilon)}$$

$\rightsquigarrow$ The lower bound is similar to prove!

Channel Coding Theorem: Achievability proof.

Let $R < C$. Let $P_X = \arg\max_{P_X} I(X;Y)$

Random Codebook $\longrightarrow \mathcal{C} = \{X^n(m)\}_{m=1}^{2^{nR}}$

<u>Joint Typicality Decoding</u>:

(ε is specified) i.i.d P_X

$$\begin{array}{l} m=1 \longrightarrow \\ m=2 \longrightarrow \\ \\ \\ \end{array} \begin{bmatrix} X_1(1) & X_2(1) & \cdots & X_n(1) \\ X_1(2) & X_2(2) & \cdots & X_n(2) \\ & \vdots & & \\ X_1(2^{nR}) & X_2(2^{nR}) & \cdots & X_n(2^{nR}) \end{bmatrix}$$

Decoder: Maximum likelihood is optimal, but we'll use suboptimal decoder.

Observe Y^n: Search for unique m such that

$$(X^n(m), Y^n) \in A_\varepsilon^{(n)}$$

Choose n large enough s.t. $P(A_\varepsilon^{(n)}) > 1 - \frac{\varepsilon}{2}$

Choose $\varepsilon < \frac{C-R}{4}$

Let $\mathcal{E}$ be the error event

$$\mathbb{P}(\mathcal{E}) = \sum_{\mathcal{C}} \Pr(\mathcal{C})\, P_e^{(n)}(\mathcal{C})$$

$$\mathbb{P}(\mathcal{E}) = \mathbb{P}[M \neq \hat{M}] = \sum_{\mathcal{C}} \Pr(\mathcal{C})\, \mathbb{P}[M \neq \hat{M} \mid \mathcal{C}]$$

$$= \sum_{\mathcal{C}} \Pr(\mathcal{C}) \cdot \frac{1}{2^{nR}} \sum_m \mathbb{P}[M \neq \hat{M} \mid \mathcal{C}, M = m]$$

$$= \frac{1}{2^{nR}} \sum_m \sum_{\mathcal{C}} \Pr(\mathcal{C})\, \mathbb{P}[M \neq \hat{M} \mid \mathcal{C}, M = m]$$

$$= \frac{1}{2^{nR}} \sum_m \sum_{\mathcal{C}} \Pr(\mathcal{C})\, \mathbb{P}[M \neq \hat{M} \mid \mathcal{C}, M = 1]$$

$$= \mathbb{P}[M \neq \hat{M} \mid M = 1]$$

Let $E_i = \{(X^n(i), Y^n) \in A_\varepsilon^{(n)}\}$ $\quad (X^n(1), Y^n) \sim P_{X^n Y^n}$

$$\mathcal{E} = E_1^c \cup E_2 \cup \cdots \cup E_{2^{nR}} \qquad X^n(j) \perp\!\!\!\perp Y^n \quad \forall j \neq 1$$

$$P(\mathcal{E} \mid M=1) \leq P[E_1^c] + \sum_{m=2}^{2^{nR}} P[E_m]$$

$$\leq \frac{\varepsilon}{2} + 2^{nR} \cdot 2^{-n(I(X;Y)-3\varepsilon)} = \varepsilon/2 + \overbrace{2^{-n(I(X;Y)-R-3\varepsilon)}}^{2^{-n\left(\frac{C-R}{4}\right)}}$$

$\longrightarrow 0$ for small enough choice of ε.

Channel Coding Theorem: Converse Proof

Let $\varepsilon > 0$ be arbitrary, let n, f, g achieve $P[M \neq \hat{M}] < \varepsilon$

$nR = H(M)$

$= I(M;\hat{M}) + \underbrace{H(M \mid \hat{M})}$ ← we'll handle using Fano

Notice: $M - X^n - Y^n - \hat{M}$,

$$I(M;\hat{M}) \leq I(X^n;Y^n) \quad \text{(Data Proc. Ineq.)}$$

$$= \sum_{i=1}^{n} I(X^n; Y_i \mid Y^{i-1})$$

$$\leq \sum_{i=1}^{n} I(X_i; Y_i \mid Y^{i-1}) + \cancel{I(X^{i-1}, X_{i+1}^n; Y_i \mid X_i, Y^{i-1})} \to 0$$

$P_{Y^n|X^n} = \prod P_{Y|X}$

$(X^n, Y^{i-1}) - X_i - Y_i$

$$\leq \sum_{i=1}^{n} I(X_i; Y_i)$$

$$= nC$$

$$\Rightarrow R \leq C + \varepsilon R + \frac{H(\varepsilon)}{n}$$

How about continuous X, Y? Then $H(X) = \infty$

$$I(X;Y) = D(P_{XY} \| P_X P_Y) = \mathbb{E}_{P_{XY}} \log\left(\frac{f_{XY}(x,y)}{f_X(x) f_Y(y)}\right)$$

$\neq H(X) - H(X|Y)$

($\infty - \infty$ problem!)

Define Differential Entropy: $h(X) = \mathbb{E}\left[\log \frac{1}{f_X(X)}\right]$

$$I(X;Y) = h(X) - h(X|Y)$$

$$A_\varepsilon^{(n)} = \left\{ x^n : \left| \frac{1}{n} \log \frac{1}{f_{X^n}(x^n)} - h(X) \right| \leq \varepsilon \right. \qquad \mathrm{Vol}(A_\varepsilon^{(n)}) \doteq 2^{n h(X)}$$

$C = \frac{1}{2} \log\left(1 + \frac{P}{N}\right)$ for Gaussian Channel